Fourier Analysis 04-27

Review.

Thm (Poisson summation formula)

Suppose $f, \hat{f} \in M(\mathbb{R})$. Then

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n x}, \quad \forall\, x\in\mathbb{R}.$$

In particular

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Example 1.

Let $\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}$, $x \in \mathbb{R}$, $t > 0$.

Heat kernel on the real line.

Notice that

$$\widehat{\mathcal{H}_t}(\xi) = e^{-4\pi^2 t \xi^2}$$

Applying Poisson summation formula gives

$$\sum_{n\in\mathbb{Z}} \mathcal{H}_t(x+n) = \underbrace{\sum_{n\in\mathbb{Z}} e^{-4\pi^2 t n^2} e^{2\pi i n x}}_{H_t(x) \text{ — heat kernel on the unit circle.}}$$

Letting $x=0$ gives

$$\sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{n^2}{4t}} = \sum_{n\in\mathbb{Z}} e^{-4\pi^2 t n^2}.$$

Prop 2: Let $f \in M(\mathbb{R})$. Suppose $\hat{f}$ is supported on $I = [-\frac{1}{2}, \frac{1}{2}]$, i.e. $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R} \setminus I$.

Then.

① The function f is determined by the values $f(n)$ at $n \in \mathbb{Z}$; more precisely

$$f(x) = \sum_{n \in \mathbb{Z}} f(n) \cdot \frac{\sin(\pi(x-n))}{\pi(x-n)}.$$

② $$\int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |f(n)|^2.$$

Pf. Notice that $\hat{f} \in M(\mathbb{R})$.

Write $g = \hat{f}$. g is supported on $[-\frac{1}{2}, \frac{1}{2}]$.

So for $x \in [-\frac{1}{2}, \frac{1}{2}]$,

$$g(x+n) = 0 \text{ for all } n \in \mathbb{Z} \setminus \{0\}.$$

Hence $$g(x) = \sum_{n \in \mathbb{Z}} g(x+n) \quad \text{for } x \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

Hence using Poisson summation formula, we have

$$g(x) = \sum_{n\in\mathbb{Z}} \hat{g}(n)\, e^{2\pi i n x}, \quad x \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

Where

$$\hat{g}(n) = \int_{-\infty}^{\infty} g(x)\, e^{-2\pi i n x}\, dx$$

$$= \int_{-\infty}^{\infty} \hat{f}(x)\, e^{-2\pi i n x}\, dx$$

$$\overset{\substack{\text{Inversion} \\ \text{formula}}}{=} f(-n).$$

Therefore

$$g(x) = \sum_{n\in\mathbb{Z}} f(-n)\, e^{2\pi i n x}.$$

$$= \sum_{n\in\mathbb{Z}} f(n)\, e^{-2\pi i n x}.$$

That is, for $x \in [-\frac{1}{2}, \frac{1}{2}]$,

$$\hat{f}(x) = \sum_{n\in\mathbb{Z}} f(n)\, e^{-2\pi i n x}.$$

From the above equation, using Parseval identity we get

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{f}(x)|^2 dx = \sum_{n \in \mathbb{Z}} |f(n)|^2$$

Since $\hat{f}$ is supported on $[-\frac{1}{2}, \frac{1}{2}]$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{f}(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx$$

$$\overset{\text{Planchrel}}{=} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

Hence

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |f(n)|^2 .$$

Next we prove ①.

Recall

$$\hat{f}(x) = \sum_{n\in\mathbb{Z}} f(n)\, e^{-2\pi i n x}, \qquad x \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

Now using Fourier inversion formula,

For all $x \in \mathbb{R}$,

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi$$

(since $\hat{f}$ is supported on $[-\tfrac{1}{2}, \tfrac{1}{2}]$)

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n\in\mathbb{Z}} f(n)\, e^{-2\pi i n \xi} \right) e^{2\pi i \xi x}\, d\xi$$

$$= \sum_{n\in\mathbb{Z}} f(n) \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i \xi (x-n)}\, d\xi$$

$$= \sum_{n \in \mathbb{Z}} f(n) \cdot \left. \frac{e^{2\pi i \xi (x-n)}}{2\pi i (x-n)} \right|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \sum_{n \in \mathbb{Z}} f(n) \cdot \frac{e^{\pi i (x-n)} - e^{-\pi i (x-n)}}{2i \ (\pi (x-n))}$$

$$= \sum_{n \in \mathbb{Z}} f(n) \cdot \frac{\sin (\pi (x-n))}{\pi (x-n)} .$$

$\blacksquare$